

Density estimates for k -impassable lattices of balls and general convex bodies in \mathbb{R}^n

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Abstract

G. Fejes Tóth posed the following problem: Determine the infimum of the densities of the lattices of closed balls in \mathbb{R}^n such that each affine k -subspace ($0 \leq k \leq n-1$) of \mathbb{R}^n intersects some ball of the lattice. We give a lower estimate for any n, k like above. If, in the problem posed by G. Fejes Tóth, we replace the ball B^n by a (centrally symmetric) convex body $K \subset \mathbb{R}^n$, we may ask for the infimum of all above infima of densities of lattices of translates of K with the above property, when K ranges over all (centrally symmetric) convex bodies in \mathbb{R}^n . For these quantities we give lower estimates as well, which are sharp, or almost sharp, for certain classes of convex bodies K . For $k = n - 1$ we give an upper estimate for the supremum of all above infima of densities, K also ranging as above (i.e., a “minimax” problem). For $n = 2$ our estimate is rather close to the conjecturable maximum. We point out the connection of the above questions to the following problem: Find the largest radius of a cylinder, with base an $(n - 1)$ -ball, that can be fitted into any lattice packing of balls (actually, here balls can be replaced by some convex bodies $K \subset \mathbb{R}^n$, the axis of the cylinder may be k -dimensional and its basis has to be chosen suitably). Among others we complete the proof of a theorem of I. Hortobágyi from 1971. Our proofs for the lower estimates of densities for balls, and for the cylinder problem, follow quite closely a paper of J. Horváth from 1970. This paper is also an addendum to a paper of the first named author from 1978 in the sense that to some arguments given there not in a detailed manner, we give here for all of these complete proofs.

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1 Preliminaries

We will work in n -dimensional Euclidean space \mathbb{R}^n . The norm of a vector x is denoted by $\|x\|$. For concepts not defined in this paper we refer, e.g., to [Gruber-Lekkerkerker \[23\]](#) and [\[42\]](#).

A set $K \subset \mathbb{R}^n$ is a *convex body* if it is convex, compact, and its *interior* $\text{int } K$ is non-empty. For $A \subset \mathbb{R}^n$ we denote by $V(A)$ its *volume* (Lebesgue measure; supposing it exists – but we will use this for bounded convex sets only), and by $\text{lin } A$, $\text{aff } A$, or $\text{conv } A$ its *linear*, *affine*, or *convex hull*, respectively. The *dimension* $\dim A$ of a convex set $A \subset \mathbb{R}^n$ is the dimension of its affine hull. $B^n \subset \mathbb{R}^n$ is the unit ball of \mathbb{R}^n with centre 0. Its volume $V(B^n)$ is denoted by κ_n . We have $\kappa_n = \pi^{n/2}/\Gamma(1 + n/2) \sim n^{-n/2}(2\pi e)^{n/2}\sqrt{\pi n}$ (by Stirling's formula). If $X \subset \mathbb{R}^n$ is a linear subspace, $X^\perp \subset \mathbb{R}^n$ denotes its *orthocomplement*. For $x, y \in \mathbb{R}^n$ we denote by $\langle x, y \rangle$ their *scalar product*. If $K \subset \mathbb{R}^n$ is a convex body with $0 \in \text{int } K$, then its *polar* (with respect to the unit sphere about 0) K^* is defined as $\{y \in \mathbb{R}^n \mid \forall x \in K \langle x, y \rangle \leq 1\}$, see [Gruber-Lekkerkerker \[23\]](#), p. 107. In our paper we will deal with *lattices* in \mathbb{R}^n , which will usually be denoted by L (and sometimes by Λ). A *cross-polytope* in \mathbb{R}^n is an affine image of the set $\text{conv}\{\pm e_1, \dots, \pm e_n\}$, where e_1, \dots, e_n is the standard base of \mathbb{R}^n .

There is a natural topology on the set of all lattices in \mathbb{R}^n . A *neighbourhood base of a lattice* $L \subset \mathbb{R}^n$ is obtained in the following way. The sets $\{\Lambda \subset \mathbb{R}^n \mid \Lambda \text{ is a lattice in } \mathbb{R}^n \text{ and has a base } \{y_1, \dots, y_n\} \text{ with } \|y_1 - x_1\| < \varepsilon, \dots, \|y_n - x_n\| < \varepsilon\}$, where $\{x_1, \dots, x_n\}$ is any (some) base of L , and $\varepsilon \in (0, \infty)$, is a neighbourhood base of L . The definitions with “any” and “some” are equivalent; in particular, the definition with “some” is independent of the choice of the base of L in question. This topology is metric and locally compact. The set of all those lattices whose minima are at least c (> 0) and the absolute values of whose determinants are at most some C ($< \infty$) is compact (for all these facts, cf. [Gruber-Lekkerkerker \[23\]](#), pp. 177–180).

def1.1 **Definition 1.1.** Let $0 \leq k \leq n - 1$ be integers, and $K \subset \mathbb{R}^n$ be a convex body. A lattice of translates of K is *k-impassable* if each affine k -subspace of \mathbb{R}^n meets some body of the body lattice. (For $k = 0$ this is the well known concept of lattice covering. For $k = 1$ or $k = n - 1$, 1-impassable or $(n - 1)$ -impassable are also called *non-transilluminable* or *non-separable*, respectively.)

R. Kannan and L. Lovász [\[32\]](#) also investigate this property and introduce, connected to this, covering minima of a convex body with respect to a lattice.

However, G. Fejes Tóth's question, to which we refer here, and the investigations of R. Kannan and L. Lovász ^[32] seem to go into, practically disjoint directions, except for Lemmas (1.2) and (2.3) in ^[32].

The main subject of this paper will be the investigation of a question of G. Fejes Tóth; see ^[17]. This is the following. Let $0 \leq k \leq n - 1$ be integers. Find the infimum of the densities of lattices of closed unit balls that are k -impassable. We will also investigate the variant when B^n is replaced by a fixed convex body $K \subset \mathbb{R}^n$. Rather up-to-date results and problems about this concept are discussed in ^[10], pp. 149-159.

def1.2 **Definition 1.2.** Let $0 \leq k \leq n - 1$ be integers. For $K \subset \mathbb{R}^n$ a convex body, $d_{n,k}(K)$ will denote the *infimum of densities of k -impassable lattices of translates of K* . If $K = B^n$, we write $d_{n,k}(B^n) = d_{n,k}$.

Evidently, $d_{n,k}(K)$, as a function of K , is affine invariant. Also, we have evidently $d_{n,0}(K) \geq \dots \geq d_{n,n-1}(K)$. Since $d_{n,0}(K)$ is the well known quantity called the density of the thinnest lattice covering by translates of K , we will investigate in general the case $1 \leq k \leq n - 1$ only.

Let $L \subset \mathbb{R}^n$ be a lattice. By $D(L)$ we denote *the absolute value of its determinant* or, what is the same, the *volume of a basic parallelotope of L* .

J. M. Wills ^[49] introduced a generalization of this concept. For $k = 1$ this concept is the well-known concept of the minimal length of a (non-0) lattice vector.

def1.3 **Definition 1.3.** (see ^[49]) Let $1 \leq k \leq n - 1$ be integers. For $L \subset \mathbb{R}^n$ a lattice, $D_k(L)$ will denote the minimum of (the absolute values of) the determinants of its k -dimensional sublattices. (Of course, we may suppose that we consider only sublattices of the form $L \cap X_k$, where X_k is a linear subspace spanned by some k vectors of L .)

def1.4 **Definition 1.4.** (H. Minkowski, e.g. ^[23], p. 58) Let $1 \leq k \leq n$ be integers, $K \subset \mathbb{R}^n$ a 0-symmetric convex body, and $L \subset \mathbb{R}^n$ a lattice. The *successive minima of K with respect to L* , denoted by $\lambda_k(K, L)$, are defined by

$$\lambda_k(K, L) = \min\{\lambda > 0 \mid \dim \text{lin}(\lambda K \cap L) \geq k\}. \quad (1) \quad \text{equal.1}$$

We will use this concept only for $K = B^n$.

def1.5 **Definition 1.5.** (Dirichlet-Voronoi) Let $A \subset \mathbb{R}^n$ be such that $\inf \{\|a_1 - a_2\| \mid a_1 \neq a_2 \in A\} > 0$ and there exists a positive number so that any closed ball in \mathbb{R}^n with that radius intersects A . Then the *Dirichlet-Voronoi cell* (*D-V cell*) of $a \in A$ with respect to A is defined as $\{x \in \mathbb{R}^n \mid \forall b \in A \setminus \{a\} \ \|x - a\| \leq \|x - b\|\}$. This is a convex polytope in \mathbb{R}^n .

We will use this concept only for A a lattice.

def1.6 **Definition 1.6.** Let $n \geq 1$ be an integer, and $K \subset \mathbb{R}^n$ be a convex body. We denote by $\delta_L(K)$, or $\vartheta_L(K)$, the density of the densest lattice packing in \mathbb{R}^n , or of the thinnest lattice covering of \mathbb{R}^n , by translates of K , respectively.

For $K = B^n$, the density $\delta_L(B^n)$ is known for $k \leq 8$ (see [Toth-1997, p. 23], and recently it has been announced for $k = 24$ by H. Cohn and A. Kumar [Cohn-Kumar1, p. 11], with an outline of proof, and proved by the same authors in [Cohn-Kumar2, p. 12], while $\vartheta_L(B^n)$ is known for $n \leq 5$ (see [Toth-1997, p. 23]). These are well-investigated quantities, and we will treat them as “known quantities”.

The following estimates are known for $\delta_L(K), \vartheta_L(K), \delta_L(B^n), \vartheta_L(B^n)$, see [Toth-1997, pp. 149-150]:

$$\frac{(\log 2 - \varepsilon)\sqrt{\pi}n^{3/2}}{4^n} \leq \delta_L(K) \leq 1 \leq \vartheta_L(K) \leq n^{(\log \log n)/\log 2 + \text{const}}, \quad (2) \quad \text{equa1.2}$$

the left hand side inequality holding for $n \geq n_\varepsilon$ (with n_ε an integer depending on ε). For centrally symmetric K the first inequality can be sharpened to

$$\frac{(\log 2 - \varepsilon)n}{2^n} \leq \delta_L(K), \quad (3) \quad \text{equa1.3}$$

for $n \geq n_\varepsilon$.

For B^n one has better estimates ([Toth-1997, p. 23], [Toth-1999, pp. 149-150], [Gruber-Lekkerkerker, p. 505], and [Rogers, p. 19]):

$$2\zeta(n)(n-1)/2^n \leq \delta_L(B^n) \leq 2^{-(0.5990\dots + o(1))n}, \quad (4) \quad \text{equa1.4}$$

$$n/(e^{3/2} + o(1)) \leq \vartheta_L(B^n) \leq \text{const} \cdot n \cdot (\log n)^{(\log(2\pi e))/\log 4}. \quad (5) \quad \text{equa1.5}$$

Our paper has a non-trivial overlap with lower density estimates of k -impassable lattices of convex bodies in [39] (in particular, their Theorem 3.1), but the results of their and our paper have been obtained independently.

2 Introduction

In this section we will recall earlier results on the subject of our paper. L. Fejes Tóth and E. Makai, Jr. [19] stated

$$d_{2,1} = \sqrt{3}\pi/8, \quad (6) \quad \text{equa2.1}$$

$$\min\{d_{2,1}(K) \mid K \subset \mathbb{R}^2 \text{ is a convex body}\} = 3/8. \quad (7) \quad \text{equa2.2}$$

In fact, they proved only that the left hand sides are at least the right hand sides, but did not prove that in the claimed cases of equalities in fact equalities hold, thus that the estimates are sharp. However, these follow from E. Makai, Jr. [38]. Namely, [38] proved

$$\min\{d_{2,1}(K) \mid K \subset \mathbb{R}^2 \text{ is a centrally symmetric convex body}\} = 1/2. \quad (8) \quad \text{equa2.3}$$

In fact, moreover [38] claimed that the minimum is attained only for a parallelogram, for which he used that

$$\min\{V(K)V(K^*) \mid K \subset \mathbb{R}^2 \text{ is 0-symmetric convex body}\} = 8 \quad (9) \quad \text{equa2.4}$$

(a theorem of K. Mahler [36]), the minimum being attained only for a parallelogram. This last statement has been proved by [41], thus completing the proof of the theorem in [38].

We have to mention that a sketch of the proof of (7) was also given in [38], see the proof of Theorem 3. For a proof of the inequality claimed and used in the proof of (7) in [38], proof of Theorem 3, we refer to [14].

R. Kannan and L. Lovász [32], Lemma (2.3), and [38], Theorem 1 proved for $K \subset \mathbb{R}^n$ a convex body

$$d_{n,n-1}(K) = \frac{V(K)V(((K-K)/2)^*)}{4^n \delta_L((K-K)/2)}. \quad (10) \quad \text{equa2.5}$$

In particular,

$$d_{n,n-1} = \frac{\kappa_n^2}{4^n \delta_L(B^n)}. \quad (11) \quad \text{equa2.6}$$

Since $\delta_L(B^n)$ is known for $n \leq 8$ (cf. [Toth-1997](#), p. 23), and for $n = 24$ has been proved in [Cohn-Kumar-2002](#), $d_{n,n-1}(B^n)$ is known for $n \leq 8$ and for $n = 24$. (Here we have to make a correction to [Makai-1978](#). There it was stated that for $n = 3$ the thinnest non-separable lattice of translates of B^3 is given by a lattice L similar to the lattice or thinnest lattice ball covering, i.e., a space-centered cubic lattice. This holds true, but the ratio of similarity was given incorrectly. L has correctly a base $\{\sqrt{2}(-1, 1, 1), \sqrt{2}(1, -1, 1), \sqrt{2}(1, 1, -1)\}$.) As a consequence, cf. [Makai-1978](#) [\[38\]](#) for $K \subset \mathbb{R}^n$ a convex body, or an 0-symmetric convex body,

$$d_{n,n-1}(K) \geq V(K)V(((K - K)/2)^*)4^{-n} \text{ or} \quad (12) \quad \text{equa2.7}$$

$$d_{n,n-1}(K) \geq V(K)V(K^*)4^{-n}, \quad (13) \quad \text{equa2.8}$$

respectively.

[Lovasz-Kannan](#) [\[32\]](#) proved

$$\begin{aligned} & \min\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is a centrally symmetric convex body}\} \geq \\ & \min\{V(K)V(K^*)4^{-n} \mid K \subset \mathbb{R}^n \text{ is a 0-symmetric convex body}\} \\ & = n^{-n}e^{O(n)}. \end{aligned} \quad (14) \quad \text{equa2.9}$$

Here for the last inequality they used the theorem of J. Bourgain-V. D. Milman, cf. [Bourgain-Milman](#) [\[9\]](#). However, this theorem was recently improved by G. Kuperberg [\[33\]](#), namely to

$$V(K)V(K^*) > \kappa_n^2/2^n. \quad (15) \quad \text{equa2.9A}$$

So, combining [equa2.9](#) [\[14\]](#) and [equa2.9A](#) [\[15\]](#), we have

$$\begin{aligned} & \min\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is a centrally symmetric convex body}\} \geq \\ & \min\{V(K)V(K^*)4^{-n} \mid K \subset \mathbb{R}^n \text{ is a 0-symmetric convex body}\} \\ & > \kappa_n^2/8^n. \end{aligned} \quad (16) \quad \text{equa2.9B}$$

However, from [equa2.7](#) [\[12\]](#) and [equa2.9A](#) [\[15\]](#), together with the observation that, for $K \subset \mathbb{R}^n$ a convex body, $V(K)V(((K - K)/2)^*) = V(K)V((K - K)/2)^{-1} \cdot V((K - K)/2)V(((K - K)/2)^*)$, where the first factor is at least $2^n \binom{2n}{n}^{-1}$ (see [Rogers](#) [\[42\]](#),

p. 37), and at most 1 (by the Brunn-Minkowski inequality, cf. [Schneider \[45\], § 6.1](#)) we have

$$\begin{aligned} & \min\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is a convex body}\} \geq \\ & \min\{V(K)V(((K-K)/2)^*)4^{-n} \mid K \subset \mathbb{R}^n \text{ is a convex body}\} > \\ & \kappa_n^2 \binom{2n}{n}^{-1} 4^{-n}. \end{aligned} \quad (17) \quad \text{equa2.10}$$

There is a counterpart to the question for the densest lattice packing of a convex body $K \subset \mathbb{R}^n$ (that is, of determining $\delta_L(K)$). Namely, one can ask for an inequality from the other side: find $\min\{\delta_L(K) \mid K \subset \mathbb{R}^n \text{ is a (centrally symmetric) convex body}\}$. These theorems are the so-called Minkowski-Hlawka type theorems; cf. [Gruber-Lekkerkerker \[23\], §19.5 Theorem 8](#), for general n and general convex bodies, and [§22, Theorem 7](#), for $n = 2$ and general convex bodies. We will need only the case $n = 2$, centrally symmetric bodies. For them we have the inequality of P. Tammela [\[47\]](#):

$$\begin{aligned} & \min\{\delta_L(K) \mid K \subset \mathbb{R}^2 \text{ is a centrally symmetric convex body}\} \\ & \geq 0.8926\dots \end{aligned} \quad (18) \quad \text{Tammela}$$

[Tammela \[47\]](#) has at its right hand side a certain, explicitly given algebraic number, which is however defined in a quite complicated way. Therefore we do not reproduce it here. However, since [Tammela \[47\]](#) is in Russian, we give hints to this definition. The last displayed formula in [\[47\]](#) is the inequality cited by us. The quantity m in it is defined by [\[47\], \(18\), \(22\)](#), as a certain rational function of u_0 , where u_0 is the unique root of the polynomial equation (23) in [\[47\]](#), in the interval given in (23) from [\[47\]](#) (the total length of these formulas is twelve lines).

Analogously, for $d_{n,k}(K)$ there is also the question of opposite, i.e., upper estimates, i.e., to a minimax problem. [Makai-1978 \[38\], Theorems 4 and 5](#) proved that

$$\begin{aligned} & \max\{d_{2,1}(K) \mid K \subset \mathbb{R}^2 \text{ is a convex body}\} = \\ & \max\{d_{2,1}(K) \mid K \subset \mathbb{R}^2 \text{ is a centrally symmetric convex body}\} \\ & \leq \pi^2/[4(3\sqrt{2} + \sqrt{3} - \sqrt{6})] = 0,6999\dots, \end{aligned} \quad (19) \quad \text{equa2.11}$$

and

$$\begin{aligned} & \max\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is convex body}\} = \\ & \max\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is a centrally symmetric convex body}\} \\ & \leq \kappa_n^2/[(n \log 2 - \text{const})2^{n-1}], \end{aligned} \quad (20) \quad \text{equa2.12}$$

for n sufficiently large.

Concerning $d_{n,k}$, besides the case $k = n - 1$ there is just one case proved by R. P. Bambah, A. C. Woods [3], namely:

$$d_{3,1} = 9\pi/32. \quad (21) \quad \boxed{\text{equa2.13}}$$

We note that the minimum is attained, e.g., for a lattice similar to the lattice of the densest lattice packing of balls in \mathbb{R}^3 . ([3] contains a small gap. In p. 153 it considers a quadratic form $G(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$, assuming integer values for all integers x_1, x_2 . It states that a, b, c are integers. However, in fact, only $a, 2b, c$ are integers. This affects case (i) in p. 153 and case (ii) in p. 154. In case (ii) we have the additional cases $a = 4, b = 1/2$, and $a = 4, b = 3/2$, both of which lead to a contradiction like in [3]. In case (i) we have the additional cases $a = 3, b = 1/2$, and $a = 3, b = 3/2$. Proceeding like in [3], we do not get a contradiction, but we obtain, as unique possibility, $c = 3$ in both of these cases. Then the determinant of (the symmetric matrix associated to) our quadratic form G is $35/4$, or $27/4$. The norm square of the primitive vector OA_1 , in p. 138, is the quotient of the determinants of the quadratic forms associated to the investigated 3-dimensional lattice, and of its 2-dimensional projection. That is, with the notations of [3], $\|OA_1\|^2 = a_{1,1}d(f)/d(G)$, where $a_{1,1} = 2$ and $d(f) = 4$, i.e., $\|OA_1\|^2$ is $64/35$, or $27/4$. However, the norm squares of the vectors of the investigated 3-dimensional lattice are even integers, a contradiction.)

Next we turn to a subject which is a bit different, when we will consider, rather than any lattices of translates of a convex body, only lattice packings of translates of a convex body. Here we will list some results for the unit ball. The relation of these two types of questions will become clear later in this paper.

A. Heppes [27] proved the following:

For each lattice packing of closed unit balls in \mathbb{R}^3 there exist three both-way infinite open circular cylinders, with linearly independent directions of axes, disjoint to the union of the ball lattice. (22) $\boxed{\text{equa2.14}}$

In fact, the proof of [Heppes](#) [\[27\]](#) shows that the radii of the bases can be $2/\sqrt{3} - 1$.
 I. Hortobágyi [\[28\]](#) stated the following sharpening of the above result:

For each lattice packing of closed unit balls in \mathbb{R}^3 there exist three both-way infinite open circular cylinders with linearly independent directions of axes, disjoint to the union of the ball lattice, with radii of their bases $3\sqrt{2}/4 - 1 = 0,0606\dots$. This inequality is sharp, even when stated for one cylinder only, for the densest lattice packing of closed unit balls. (23) equa2.15

[Hortobágyi](#) [\[28\]](#) did not prove that in the claimed case of equality in fact equality holds (even when stated for one cylinder only), thus that his estimate is sharp. We will see that the result of [Bambah-Woods](#) [\[3\]](#) and our Theorem 3.25 together will prove that in the above claimed case of equality in fact equality holds; not even one such cylinder exists with a greater radius of its base.

J. Horváth [\[29\]](#), Satz 1, and J. Horváth and S. S. Ryskov [\[31\]](#) proved an analogous result for \mathbb{R}^4 :

For each lattice packing of closed unit balls in \mathbb{R}^4 there exist four both-way infinite open cylinders, disjoint to the union of the ball lattice, with linearly independent directions of axes and with bases that are 3-balls of radii $\sqrt{5}/2 - 1 \geq 0,1180\dots$. (24) equa2.16

(We note that still in [Horváth-1970](#) [\[29\]](#), sharpness of the result was claimed, and this was repeated in [Ryskov-Horváth](#) [\[43\]](#), p. 128; however, this was withdrawn in [Horváth-Ryskov-1975](#) [\[31\]](#), available only in Hungarian, in p. 92, (2), where already only “ $R_4 \geq \sqrt{5}/2 - 1$ ” stands, together with an explanation in p. 94, Paragraph 2, end of last line, that “ R_4 is probably sharp”.)

A critic of [Horváth-1970](#) [\[29\]](#), Satz 2, is contained in T. Hausel [\[24\]](#). He shows:

There exist lattice packings of closed balls in \mathbb{R}^n such that for $k \geq n - c\sqrt{n}$ ($c > 0$ some constant) they are k -impassable. (25) equa2.17

This was sharpened by [Henk-Zieler-Zowa](#) [\[26\]](#), or [\[25\]](#), who showed that

in the above statement one may replace the hypothesis $k \geq n - c\sqrt{n}$ by $k \geq n - cn$ ($c > 0$ some constant), or by n is sufficiently large, and $k \geq n/\log_2 n$, and still the same conclusion holds. (26) equa2.18

3 Results

First we determine $d_{n,n-1}(K)$ for a simplex. (This was claimed, but not proved, in [Makai-1978](#), Proposition 1. The statements there, not included in our Proposition 1 here, easily follow from our proof here.)

Proposition 3.1. *Let $n \geq 2$, and let $K \subset \mathbb{R}^n$ be an n -simplex. Then we have*

$$d_{n,n-1}(K) = \frac{n+1}{2^n n!} = n^n \left(\frac{e}{2}\right)^n e^{o(n)}.$$

Definition 3.2. ([Wills](#), [\[49\]](#)) Let $1 \leq k \leq n-1$ be integers, and $L \subset \mathbb{R}^n$ be a lattice. We denote by $D_k(L)$ the quantity $\min\{|\det L_k| \mid L_k \text{ is a } k\text{-dimensional sublattice of } L\}$. (Evidently we may assume additionally that $L_i = L \cap \text{lin } L_i$; this does not affect the value of the minimum.) For $k = d$ we write $D(L) = D_d(L)$. For $d = 1$, we have that $D_1(L)$ is the minimal length of a non-0 vector in L .

The existence of the minimum was stated both in [Wills](#), [\[49\]](#), p. 268, and [Schnel](#), [\[46\]](#), p. 607. It follows from the version of the selection theorem of Mahler ([\[23\]](#), p. 179) for k -dimensional lattices in \mathbb{R}^n , for which the proof in [Gruber-Lekkerkerker](#), [\[23\]](#), pp. 179-180, goes through without any change. Then the hypotheses of this version of Mahler's Theorem are satisfied. 1) Each k -dimensional sublattice of L has as minimum at least that of L . 2) Looking for the infimum of the absolute values of the k -dimensional determinants of the k -dimensional sublattices, we may assume that we consider only such k -dimensional sublattices for which these absolute values of k -dimensional determinants are bounded from above.

Definition 3.3. Let $1 \leq k \leq n-1$ be integers. We denote by $c_{n,k}$ the number $\min\{c \in (0, \infty) \mid \text{for each lattice } L \subset \mathbb{R}^n \text{ we have } D_k(L) \leq c \cdot D(L)^{k/n}\}$.

In Theorem 3.5 we will see that the set of numbers c in question is nonempty. Then the existence of the minimum of the numbers c is obvious.

We remark that $c_{n,1}$ is a well-known quantity: we have

$$\left(\frac{c_{n,1} D(L)^{1/n}}{2}\right)^n \kappa_n / D(L) = \delta_L(B^n).$$

From here and from the estimates in § 1 we have

$$\sqrt{n} \left(\sqrt{\frac{2}{\pi e}} \cdot \frac{1}{2} - o(1) \right) \leq c_{n,1} \leq \sqrt{n} \left(\sqrt{\frac{2}{\pi e}} \cdot \frac{1}{2^{0,5990\dots}} + o(1) \right).$$

On the other hand, ^{Schnell}~~[46]~~, Theorem 3, says

$$c_{n,n-1} \leq (n^{3/2}/2)\kappa_n^{1/n} \sim \sqrt{\pi e/2} \cdot n, \quad (27) \quad \boxed{\text{equa3.1}}$$

which is much better than our following theorem, applied to the case $k = n - 1$. (Also, ^{equa3.1}~~(27)~~ substantially improves ^{Willis}~~[49]~~, Theorem 3.) Our Theorem 3.5 will be good for not too large values of k , cf. Remark 3.6. If we would have defined also $c_{n,n}$, analogously we would have $c_{n,n} = 1$. Anyway, for each $k \in \{1, \dots, n - 1\}$ we have $c_{n,k} \geq 1$. In fact, for $L = \mathbb{Z}^n$ we have $D_k(\mathbb{Z}^n)$ a positive integer. And, since $\mathbb{Z}^k \subset \mathbb{Z}^n$, actually $D_k(\mathbb{Z}^n) = 1$. Observe still that in the next theorem the $(1/k)$ -th power of the upper estimate for $c_{n,k}$ is independent of k .

Problem 3.4. Determine as many $c_{n,k}$'s as possible.

Theorem 3.5. *Let $1 \leq k \leq n - 1$ be integers. Then the numbers $c_{n,k}$ from Definition 3.2 satisfy*

$$c_{n,k} \leq 2^k \left(\frac{\delta_L(B^n)}{\kappa_n} \right)^{k/n} \leq 2^k \left(2^{-0,5990\dots} \cdot \sqrt{\frac{n}{2\pi e}} (1 + o(1)) \right)^k.$$

For $k = 1$ the first inequality is an equality, while for $2 \leq k \leq n - 1$ it is a strict inequality.

The question whether $d_{n,k}$, or $d_{n,k}(K)$, respectively, is a minimum is not clear. For $k = 0$ we have the question of existence of the thinnest lattice covering with translates of B^n or K , respectively, which is known to exist. For $k = n - 1$, by ^{Lovász-Kannan-1978}~~[32]~~ and ~~[38]~~ a lattice $\{x + K \mid x \in L\}$ of translates of a convex body K is non-separable if and only if the body lattice $\{x^* + (((K - K)/2)^*)/4 \mid x^* \in L^*\}$, where L^* is the lattice polar to L , is a lattice packing, and the product of the densities of these two body lattices is independent of L (it depends on K only). Since densest lattice packings of a convex body exist, there exist also thinnest non-separable lattices of a convex body.

However, for $1 \leq k \leq n - 2$ the situation seems to be different. We have

Proposition 3.6. *Let $1 \leq k \leq n - 2$ be an integer, $K \subset \mathbb{R}^n$ a strictly convex body, and $X \subset \mathbb{R}^n$ a k -dimensional linear subspace. Then the set of lattices $L \subset \mathbb{R}^n$ such that for the body lattice $\{x + K \mid x \in L\}$ there does not exist a translate of X disjoint to $\bigcup\{x + K \mid x \in L\}$ is not closed in the topology of lattices described in § 1. Moreover, there exists a lattice L such that there is a translate of X disjoint to $\bigcup\{x + K \mid x \in L\}$, but for any $\varepsilon > 0$ there does not exist any translate of X disjoint to $\bigcup\{x + K \mid x \in (1 - \varepsilon)L\}$ (and here $(1 - \varepsilon)L$ converges to L for $\varepsilon \rightarrow 0$).*

For the quantity $d_{n,k}$, defined in Definition 1.2, we have

Theorem 3.7. *Let $1 \leq k \leq n - 1$ be integers. Then we have*

$$d_{n,k} \geq \frac{\kappa_n \vartheta_L(B^{n-k})^{n/(n-k)}}{\kappa_{n-k}^{n/(n-k)} c_{n,k}^{n/(n-k)}} \geq \frac{\kappa_n \vartheta_L(B^{n-k})^{n/(n-k)}}{\kappa_{n-k}^{n/(n-k)} 2^{kn/(n-k)} (\delta_L(B^n)/\kappa_n)^{k/(n-k)}}.$$

For $k \geq 2$ the second inequality is strict. For $k = 1$, in the inequality, obtained from the above chain of inequalities, by omitting the middle term, we have equality for $n = 2$, and we have strict inequality for $3 \leq n \leq 6$. We have also

$$d_{n,k} \geq \frac{\kappa_n^2}{4^n \delta_L(B^n)} = \frac{e^{O(n)}}{n^n}.$$

Remark 3.8. The smaller lower estimate in the first chain of inequalities in Theorem 3.7 can be still (not substantially) diminished by using $\vartheta_L(B^{n-k}) \geq (n - k)e^{-3/2}(1 + o(1))$, and $\delta_L(B^n) \leq 2^{-(0.5990\dots + o(1))n}$, and then by Stirling's formula we obtain, for $n \rightarrow \infty$ and $n - k \rightarrow \infty$,

$$d_{n,k} \geq \left((1/(\sqrt{n})^{k/(n-k)}) \left(\sqrt{\pi e/2} \cdot 2^{0.5990\dots + o(1)} \right)^{k/(n-k)} \sqrt{(n-k)/n} \right)^n \times ((n-k)e^{-3/2})^{n/(n-k)} ((n-k)/n)^{n/(n-k)}.$$

Here the factors are written in the order according to their “contribution to the product”. Thus one sees, e.g., that for $k \geq 1$ fixed, and $n \rightarrow \infty$, this behaves approximately like $\text{const}_k \cdot n/(\sqrt{n})^k$. If $k/n = c < 2/3$ is fixed, it

behaves like $e^{O(n)}/n^{nc/(2-2c)}$, where $c/(2-2c) < 1$, which is still a better estimate than $\kappa_n^2/(4^n \delta_L(B^n)) = n^{-n} e^{O(n)}$. We yet remark that, for $k \geq n-1$, the first inequality of Theorem 3.7 and (27) give $d_{n,n-1} \geq n^{-3n/2} e^{O(n)}$, rather than the correct $d_{n,n-1} = n^{-n} e^{O(n)}$, cf. (11).

Since $\delta_L(B^n)$ is known for $n \leq 8$ and $n = 24$, and $\vartheta_L(B^n)$ is known for $n \leq 5$, we can evaluate the second lower bound in Theorem 3.7 for $n \leq 8$ and $n = 24$, and for $n - k \leq 5$. Since the second lower bound in Theorem 3.7 is sharp only for $k = 1$, we can have sharp results from Theorem 3.7 only for $k = 1$, and thus $n \leq 6$.

Observe that for the case $n = 2$ the sharp estimate $d_{2,1} = \sqrt{3}\pi/8$ was given already in [19], also cf. [38]. As concerns the case $k = 1$ and $n \geq 3$, in [31], p. 92, (2), it was stated, without proof, that for $n = 5, 6$ the projection of the (unique; see [23], p. 517) densest lattice packings of balls in \mathbb{R}^n , when projected along any minimum vector, will not yield a lattice similar to a lattice of thinnest lattice coverings of balls in \mathbb{R}^{n-1} . We will give a proof of this for “some minimum vector” in the proof of Theorem 3.7, which will imply the non-sharpness of the estimates mentioned in Theorem 3.7, for $n = 5, 6$. (For $n = 5$ the minimum vectors are equivalent under the group of congruences of the respective lattice.) For $n = 3$ the analogous statement is simple; cf. in the proof of Theorem 3.7. For $n = 4$ we do have similarity of the two lattices (the projection of a space centred cubic lattice along the direction of a coordinate axis is a space centred cubic lattice, and all minimum vectors of the space centred cubic lattice in \mathbb{R}^4 are equivalent under the group of congruences of this lattice). However, a suitable projection in a direction different from the directions of all minimum vectors will prove the needed non-sharpness of the estimate mentioned in Theorem 3.7, for $n = 4$.

We give an upper estimate for $d_{n,k}$, for certain k 's, which is probably rather weak.

Proposition 3.11. *There exists a constant n_0 such that the following holds. For any integers $n \geq n_0$ and $n/\log_2 n \leq k \leq n-1$ we have $d_{n,k} \leq \delta_L(B^n) \leq 2^{-(0.5990+o(1))n}$.*

Now we turn from B^n to arbitrary (centrally symmetric) convex bodies. We use $c_{n,k}$ from Definition 3.3.

Theorem 3.12. *Let $1 \leq k \leq n-1$ be integers. Then we have*

$$\begin{aligned} \min\{d_{n,k}(K) \mid K \subset \mathbb{R}^n \text{ is a convex body}\} &\geq \\ &\frac{\frac{d_{n,k}}{\kappa_n n! n^{n/2}/(n+1)^{(n+1)/2}}}{\frac{\kappa_n \vartheta_L(B^{n-k})^{n/(n-k)} / \left(\kappa_{n-k}^{n/(n-k)} c_{n,k}^{n/(n-k)} \right)}{\left[\kappa_n n! n^{n/2}/(n+1)^{(n+1)/2} \right]}} \geq \\ &\frac{\kappa_n \vartheta_L(B^{n-k})^{n/(n-k)} / \left(\kappa_{n-k}^{n/(n-k)} 2^{kn/(n-k)} (\delta_L(B^n)/\kappa_n)^{k/(n-k)} \right)}{\kappa_n n! n^{n/2}/(n+1)^{(n+1)/2}}. \end{aligned}$$

Further, we have

$$\begin{aligned} \min\{d_{n,k}(K) \mid K \subset \mathbb{R}^n \text{ is a centrally symmetric convex body}\} &\geq \\ &\frac{\frac{d_{n,k}}{\kappa_n n!/2^n}}{\frac{\kappa_n \vartheta_L(B^{n-k})^{n/(n-k)} / \left(\kappa_{n-k}^{n/(n-k)} c_{n,k}^{n/(n-k)} \right)}{\kappa_n n!/2^n}} \geq \\ &\frac{\kappa_n \vartheta_L(B^{n-k})^{n/(n-k)} / \left[\kappa_{n-k}^{n/(n-k)} 2^{kn/(n-k)} (\delta_L(B^n)/\kappa_n)^{k/(n-k)} \right]}{\kappa_n n!/2^n}. \end{aligned}$$

Also we have

$$\min\{d_{n,k}(K) \mid K \subset \mathbb{R}^n \text{ is a convex body}\} \geq \frac{\kappa_n^2}{\binom{2n}{n}} = \frac{e^{O(n)}}{n^n}.$$

Remark 3.13 The denominators of the fractions in Theorem 3.12 in square brackets are $n^{n/2}e^{O(n)}$. Hence, for $k/n = c < 1/2$ fixed, the smaller lower estimate in the chain of inequalities in Theorem 3.7 behaves “approximately” like $1/n^{c/(2-2c)}$, where $c/(2-2c) < 1/2$. Hence we get still a better estimate than $n^{-n}e^{O(n)}$.

Still we remark that for general n, k we have no conjecture about the infima of the quantities investigated in Theorem 3.12. However, for $k = n-1$ we do have (for the existence of the minima, cf. the proof of Theorem 3.16):

Conjecture 3.14. We have

$$\min\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is a convex body}\} = \frac{n+1}{2^n n!},$$

and

$$\min\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is a centrally symmetric convex body}\} = \frac{1}{n!}.$$

If true, these would be sharp, cf. our Proposition 3.1 and [Makai-1978](#) [\[38\]](#), Proposition 2, which show that in the first case equality holds for a simplex, and in the second case for a cross-polytope.

Since by [Makai-1978](#) [\[38\]](#), Theorem 1, we have $V(K)V((K-K)/2)^*/[4^n \delta_L(((K-K)/2)^*)] \geq V(K)V((K-K)/2)^*/4^n$, the centrally symmetric case (when $((K-K)/2)^* = K^*$) would follow from the volume product conjecture of K. Mahler (i.e., that for 0-symmetric convex bodies $K \subset \mathbb{R}^n$ we have $V(K)V(K^*) \geq 4^n/n!$); see [Mahler39](#) [\[36\]](#).

Some particular cases of the centrally symmetric case can be proved.

Definition 3.15. ([Goodey-Weil](#) [\[20\]](#), Goodey-Weil) We call a convex body $K \subset \mathbb{R}^n$ a *zonoid* if it is the Hausdorff limit of some $K_i \subset \mathbb{R}^n$, where each K_i is a *zonotope*, i.e., a finite vector sum of line segments. (Of course, each zonoid is centrally symmetric.)

For a 0-symmetric convex body $K \subset \mathbb{R}^n$, the *associated norm* is the norm on \mathbb{R}^n whose unit ball is K .

Theorem 3.16. *Let $n \geq 3$ be an integer. We have $\min\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is a zonoid}\} > \min\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is a polar of a zonoid centred at } 0\} = \min\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is symmetric w.r.t. all coordinate hyperplanes}\} = \min\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is 0-symmetric, and the norm associated to } K \text{ satisfies that all the natural projections to the coordinate hyperplanes are contractions}\} = 1/n!$. For $n \leq 8$ we have also $\min\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is a 0-symmetric convex polytope with at most } 2n+2 \text{ facets}\} > \min\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is a 0-symmetric convex polytope with at most } 2n+2 \text{ vertices}\} = 1/n!$. The second, third and sixth minima are attained only for cross-polytopes.*

For the general case we have an analogous

Conjecture 3.17. For $K \subset \mathbb{R}^n$ a convex body we have

$$V(K)V(((K-K)/2)^*) \geq \frac{2^n(n+1)}{n!},$$

possibly with equality only for a simplex. (For a simplex we have equality, cf. the proof of Proposition 3.1.)

The case $n = 2$, including the case of equality (i.e., that it occurs only for the triangle) is proved by H. G. Eggleston [14]. Observe that

$$\begin{aligned} V(K)V(((K-K)/2)^*) = \\ [V(K)/V((K-K)/2)] \cdot [V((K-K)/2) \cdot V(((K-K)/2)^*)] \geq \\ [2^n/\binom{2n}{n}] \cdot \min\{V(K)V(K^*) \mid K \subset \mathbb{R}^n \text{ is a 0-symmetric convex body}\}. \end{aligned}$$

Here we used the difference body inequality; see [Rogers 42], p. 37. If Mahler's volume product conjecture would be true, this quantity would be $\sim [2^n(n+1)/n!]\sqrt{\pi/n}$, thus very close to the conjectured value. Anyway, by [H7], $V(K)V(((K-K)/2)^*) \geq \kappa_n^2 \binom{2n}{n}^{-1}$. This remark hints to that the proof of Conjecture 3.17 would be quite difficult (as is the case with the sharp lower estimate in the volume product problem).

Analogously like above, in a particular case we have an almost sharp estimate.

Theorem 3.18. *We have $2^n(n+1)/n! \geq \min\{d_{n,n-1}(K) \mid K \subset \mathbb{R}^n \text{ is a convex body such that } ((K-K)/2)^* \text{ is a zonoid}\} \geq [2^n/\binom{2n}{n}]/n! \sim [2^n(n+1)/n!]\sqrt{\pi/n}$.*

Another way of approach for general convex bodies would be the following by [Makai-1978 38], Corollary 1. For a lattice L and a convex body K the body lattice $\{K+x \mid x \in L\}$ is $(n-1)$ -impassable if and only if the body lattice $\{(K-K)/2+x \mid x \in L\}$ is $(n-1)$ -impassable. Then include $(K-K)/2$ into an (0-symmetric) ellipsoid E such that $V(E)/V(K)$ would be possibly small. Then also $\{E+x \mid x \in L\}$ is $(n-1)$ -impassable. We have $d_{n,n-1}(K)/V(K) = d_{n,n-1}((K-K)/2)/V((K-K)/2)$, and the ratio of the densities of $\{K+x \mid x \in L\}$ and $\{E+x \mid x \in L\}$ is $V(K)/V(E)$. Further, the density of $\{E+x \mid x \in L\}$ is at least $d_{n,n-1}$. So

$$\begin{aligned} d_{n,n-1}(K) \geq \\ d_{n,n-1}/\min\{V(E)/V(K) \mid E \subset \mathbb{R}^n \text{ is an (0-symmetric) ellipsoid,} \\ (K-K)/2 \subset E\}. \end{aligned}$$

Conjecture 3.19. For each convex body $K \subset \mathbb{R}^n$ there exists an ellipsoid $E \subset \mathbb{R}^n$ such that $(K-K)/2 \subset E$, and $V(E)/V(K)$ is at most the same quantity when K is a regular simplex and E is the circumball of $(K-K)/2$, i.e., a ball of the same diameter as K . In other words, $V(E)/V(K) \leq \kappa_n n!/(2^{n/2}\sqrt{n+1})$.

$\pi\sqrt{2}$, then we would have $\min\{d_{3,2}(K) \mid K \subset \mathbb{R}^3 \text{ is a convex body}\} \geq (\pi/(6\sqrt{2})) / (\pi\sqrt{2}) = 1/12$, as stated in Conjecture 3.14 for $n = 3$.

All these point to that for finding $\min\{d_{3,2}(K) \mid K \subset \mathbb{R}^3 \text{ is a convex body}\}$, the way through Conjecture 3.19 would be a more realistic plan than going through Conjecture 3.17.

Remark 3.21. Now we compare the values in Conjecture 3.14 and those following from Theorem 3.12, first the inequalities in both chains of inequalities, for $n = 3, \dots, 8, 24$, $k = n - 1$ (when $d_{n,n-1}$ is known, cf., e.g., [42], p. 3, and [11], [12]).

For the general case the values in Conjecture 3.14 are, in the above order, 0,08335..., 0,01302..., 0,001562..., 0,0001519..., 0,00001240..., 0,0000008717..., $2,402 \cdot 10^{-30}$, while for the centrally symmetric case they are, in the above order, 0,1667..., 0,04167..., 0,008333..., 0,001389..., 0,0001984..., 0,00002480..., $1,612 \cdot 10^{-24}$ (for $n = 24$ cf. also [23], p. 522, and [34]).

The first inequalities in both chains of inequalities in Theorem 3.12 give, for the general case, in the above order 0,04538..., 0,004548..., 0,0003558..., 0,00001974..., 0,0000008751..., 0,00000002909..., $4,673 \cdot 10^{-38}$, while for the centrally symmetric case they are, in the above order, 0,1179..., 0,02083..., 0,002947..., 0,0003007..., 0,00002482..., 0,000001550..., $9,607 \cdot 10^{-32}$. These mean that our estimates for $3 \leq n \leq 8$ can be considered as relatively good.

Now we will sharpen (II9) in § 2. Similarly as there is a counterpart to the question for the densest lattice packing of a convex body $K \subset \mathbb{R}^n$ (that is, of determining $\delta_L(K)$), one can ask for an inequality from the other side: find $\min\{\delta_L(K) \mid K \subset \mathbb{R}^n \text{ is a (centrally symmetric) convex body}\}$. This is the so-called Minkowski-Hlawka theorem, with its variants. In our case we are interested in $\min\{d_{n,k}(K) \mid K \subset \mathbb{R}^n \text{ is a (centrally symmetric) convex body}\}$. But here we have also a counterpart: find $\max\{\delta_L(K) \mid K \subset \mathbb{R}^n \text{ is a (centrally symmetric) convex body}\}$. This is a minimax problem. Like for the Minkowski-Hlawka theorem, for general n there does not seem to be a simple answer, what the extremal bodies would be, like for general n and k . For $k = 0$ our question reduces to finding $\max\{\delta_L(K) \mid K \subset \mathbb{R}^n \text{ is a (centrally symmetric) convex body}\}$. For $k = 0$ and $n = 2$ the solution is known, both for the general and for the centrally symmetric case: K is a triangle, or an ellipse, respectively (cf. [23], p. 249, Theorem of I. Fáry, and p. 247, Theorem of L. Fejes Tóth, R. P. Bambah, and C. A. Rogers).

The next interesting case is $n = 2, k = 1$. We improve [\(19\)](#) in § 2, in such a way, that the difference between the proved and the conjectured upper bound will be reduced by a factor about 2. [equa2.11](#)

Theorem 3.22. *We have $\max\{d_{2,1}(K) \mid K \subset \mathbb{R}^2 \text{ is a convex body}\} = \max\{d_{2,1}(K) \mid K \subset \mathbb{R}^2 \text{ is a centrally symmetric convex body}\} \leq 0,6910\dots$*

We note that the inequality of [\[47\]](#) has at its right hand side a certain, explicitly given algebraic number, which is however defined in a quite complicated way. Therefore we do not reproduce it here. However, since [\[47\]](#) is in Russian, we give hints to this definition. The last displayed formula in [\[47\]](#) is the inequality cited by us. The quantity m in it is defined by [\[47\]](#), (18), (22), as a certain rational function of u_0 , where u_0 is the unique root of the polynomial equation (23) in [\[47\]](#), in the interval given in (23) from [\[47\]](#) (the total length of these formulas is twelve lines).

Conjecture 3.23. Among the convex bodies $K \subset \mathbb{R}^2$, the quantity $d_{2,1}(K)$ attains its maximum for a circle, i.e., is equal to $\sqrt{3}\pi/8 = 0,6802\dots$.

Below (Remark 3.26) we show how R. P. Bambah's and A. C. Woods' Theorem (see [\[3\]](#)) implies Hortobágyi's Theorem (cf. [\[28\]](#)) (however, without the statement that there exist even three linearly independent directions with the stated property; i.e., his statement for one direction only). At the same time we sharpen the Theorem of Horváth (see [\[29\]](#), Satz I); also in this case without the statement that there exist even four linearly independent directions with the stated property, i.e., his statement for one direction only). We have to remark that this theorem was claimed in [\[29\]](#) to be sharp. In a paper in Hungarian ([\[31\]](#), pp. 91 and 94) he has withdrawn this claim. Below we will see that this claim does not hold; cf. our Proposition 3.27 below. Rather than B^n , we will consider any convex body $K \subset \mathbb{R}^n$.

Theorem 3.25. *Let $1 \leq k \leq n - 1$ be integers and $K \subset \mathbb{R}^n$ a convex body. Suppose $\delta_L(K) < d_{n,k}(K)$. Then for any lattice packing $\{K + x \mid x \in L\}$ of translates of K there exists an affine k -plane $A_k \subset \mathbb{R}^n$ such that $\text{int}(A_k + [(d_{n,k}(K)/\delta_L(K))^{1/n} - 1](-K))$ is disjoint to $\bigcup\{K + x \mid x \in L\}$.*

Remark 3.26. A. Bambah-Woods' Theorem (see [\[3\]](#)) asserts $d_{3,1} = 9\pi/32$, with equality, e.g., for the lattice generated by $(4/3)(0, 1, 1), (4/3)(1, 0, 1), (4/3)(1, 1, 0)$. We have $\delta_L(B^3) = \pi/(3\sqrt{2})$ (see [\[42\]](#), p. 3). That is smaller than $d_{3,1} = 9\pi/32$. Then Theorem 3.25 implies that for any lattice packing of unit balls in \mathbb{R}^3 there is an open, both-way infinite cylinder with base a circle of radius $(d_{3,1}/\delta_L(B^3))^{1/3} - 1 = [(9\pi/32)/(\pi/3\sqrt{2})]^{1/3} - 1 = 3\sqrt{2}/4 - 1$, [Rogers](#)

disjoint to our lattice packing of unit balls. That equals the value of the radius in [28], Satz 1, cf. our (2.15) in § 2, hence we obtained a new proof of the inequality in [28], Satz 1. (We remind once more that we obtained this way just one cylinder of this radius, while [28], Satz 1, obtained three ones with linearly independent axis directions.)

B. Now we show that Bambah-Woods' Theorem implies that the value of the radius $3\sqrt{2}/4 - 1$ is sharp in Hortobágyi's Theorem, even when stated for one direction only — namely for the densest lattice packing of unit balls in \mathbb{R}^3 — since that was claimed but not proved there. Suppose that there exists an open both-way infinite cylinder of radius $r > 3\sqrt{2}/4 - 1$, disjoint to $\bigcup\{B^3 + x \mid x \in L\}$ — where L is the point lattice corresponding to the densest lattice packing of unit balls. Then the axis of this cylinder is disjoint to $\bigcup\{(r + 1)\text{int } B^3 + x \mid x \in L\}$. By $r + 1 > 3\sqrt{2}/4$, this last lattice of open balls has a density $\delta_L(B^3)(r + 1)^3 = [\pi/(3\sqrt{2})](r + 1)^3 > [\pi/(3\sqrt{2})](3\sqrt{2}/4)^3 = 9\pi/32 = d_{3,1}$. Now, replacing $(r + 1)\text{int } B^3$ by $(3\sqrt{2}/4)B^3$, the lattice $\{(3\sqrt{2}/4)B^3 + x \mid x \in L\}$ has a density equal to $d_{3,1}$, and its complement contains a line. However, by [3], taking a lattice packing of closed balls of some radius, the lattice being that of the densest packing, and the density being $d_{3,1}$, the complement cannot contain a line. This is a contradiction, so $r > 3\sqrt{2}/4 - 1$ cannot happen, which means that Hortobágyi's Theorem is sharp.

In a way very similar to that from **A**, our Theorem 3.25 implies the following proposition, that sharpens the Theorem Horvath ([29], Satz I), and [31], cf. our (2.4) in § 2, although with a not explicit constant. We remind once more that we obtain this way just one cylinder, although with larger radius of base, while [29], Satz I, obtained four ones with linearly independent axis directions.

Proposition 3.27. *For each lattice packing of closed unit balls in \mathbb{R}^4 there exists a both-way infinite open cylinder, disjoint to the union of the ball lattice, with base that is a 3-ball of radius at least some number $c > \sqrt{5}/2 - 1 \geq 0,1180\dots$*

4 Proofs

Proof of Proposition 3.1: Since $d_{n,k}(K)$ is affine invariant, we suppose that K is a regular simplex S , say, with edge length $\sqrt{2}$. We embed S in \mathbb{R}^{n+1} as $\text{conv}\{e_1, \dots, e_n, e_{n+1}\}$, where e_i are the usual basis vectors. The

projection of $\{(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1\}$ to the x_1, \dots, x_n -coordinate plane is bijective, and the vertices of S project to $e_1, \dots, e_n, 0$, respectively. The inhomogeneous lattice in \mathbb{R}^n generated by these projections is $\mathbb{Z}^n \subset \mathbb{R}^n$, so the inhomogeneous lattice generated by e_1, \dots, e_n, e_{n+1} in \mathbb{R}^{n+1} is its inverse image by this projection, i.e., $\{(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1, (x_1, \dots, x_n) \in \mathbb{Z}^n\} = \{(j_1, \dots, j_n, j_{n+1}) \in \mathbb{Z}^{n+1} \mid \sum_{i=1}^{n+1} j_i = 1\}$.

Let us consider the lattice translate of the last considered inhomogeneous lattice, i.e., $L := \{(j_1, \dots, j_n, j_{n+1}) \in \mathbb{Z}^{n+1} \mid \sum_{i=1}^{n+1} j_i = 0\}$. We will determine the D-V cell of 0 with respect to L , taken in the linear hull $\text{lin } L$ of L .

Observe that the minimal length of a non-0 vector of L is $\sqrt{2}$, and is attained exactly for $e_j - e_l$, $1 \leq j, l \leq n+1$. (Namely, any non-0 vector of L has at least two non-0 coordinates, and their absolute values are at least 1. If its length is $\sqrt{2}$, the above two non-0 coordinates have absolute value 1, and all other coordinates are 0.)

The D-V cell of 0 with respect to L , considered in $\text{lin } L$, is contained in $\{x = (x_1, \dots, x_n, x_{n+1}) \in \text{lin } L \mid \|x\|^2 \leq \|x + e_j - e_l\|^2 \text{ } (1 \leq j \neq l \leq n+1)\}$. We have

$$\begin{aligned} \|x\|^2 \leq \|x + e_j - e_l\|^2 &\Leftrightarrow \langle x, x \rangle \leq \langle x + e_j - e_l, x + e_j - e_l \rangle \Leftrightarrow \\ 0 \leq 2\langle x, e_j - e_l \rangle + \langle e_j - e_l, e_j - e_l \rangle &\Leftrightarrow 0 \leq \langle x, e_j - e_l \rangle + 1 \Leftrightarrow \\ x_l - x_j \leq 1. \end{aligned}$$

Hence these inequalities hold for all $1 \leq j, l \leq n+1$ if and only if

$$\max_{1 \leq j \leq n+1}$$

$x_j - \min_{1 \leq j \leq n+1} x_j \leq 1$. So, any x in the D-V cell of 0 with respect to L , considered in $\text{lin } L$, satisfies the last inequality. Thus, any x in the relative interior, with respect to $\text{lin } L$, of this D-V cell satisfies $\max_{1 \leq j \leq n+1} x_j - \min_{1 \leq j \leq n+1} x_j < 1$.

This also implies that any x in the relative interior, with respect to $\text{lin } L$, of the D-V cell of $(i_1, \dots, i_n, i_{n+1}) \in L$ with respect to L satisfies $\max_{1 \leq j \leq n+1} (x_j - i_j) - \min_{1 \leq j \leq n+1} (x_j - i_j) < 1$. Let $C(i_1, \dots, i_n, i_{n+1})$, or $D(i_1, \dots, i_n, i_{n+1})$ denote the relative interior, with respect to $\text{lin } L$, of the D-V cell of $(i_1, \dots, i_n, i_{n+1}) \in L$, considered in $\text{lin } L$, or $\{(x_1, \dots, x_n, x_{n+1}) \in \text{lin } L \mid \max_{1 \leq j \leq n+1} (x_j - i_j) - \min_{1 \leq j \leq n+1} (x_j - i_j) < 1\}$, respectively. Since we know that $C(0, \dots, 0, 0) \subset D(0, \dots, 0, 0)$, we have for each $(i_1, \dots, i_n, i_{n+1}) \in L$ that $C(i_1, \dots, i_n, i_{n+1}) \subset D(i_1, \dots, i_n, i_{n+1})$.

We assert that the sets $D(i_1, \dots, i_n, i_{n+1})$, for $(i_1, \dots, i_n, i_{n+1}) \in L$, are disjoint. Of course, it is sufficient to show $D(0, \dots, 0, 0) \cap D(i_1, \dots, i_n, i_{n+1}) =$

\emptyset , for $(0, \dots, 0, 0) \neq (i_1, \dots, i_n, i_{n+1})$. We have $\min_{1 \leq j \leq n+1} i_j \leq -1$ or $\max_{1 \leq j \leq n+1} i_j \geq 1$. Since $\sum_{j=1}^{n+1} i_j = 0$, any of these inequalities implies the other one (e.g., if $\max_{1 \leq j \leq n+1} i_j \geq 1$ and $\min_{1 \leq j \leq n+1} i_j \geq 0$, then $\sum_{j=1}^{n+1} i_j \geq 1$; the other case is similar). So we have both $\min_{1 \leq j \leq n+1} i_j \leq -1$ and $\max_{1 \leq j \leq n+1} i_j \geq 1$. Now suppose $(x_1, \dots, x_n, x_{n+1}) \in D(0, \dots, 0, 0) \cap D(i_1, \dots, i_n, i_{n+1})$, and, e.g., $i_1 \leq -1, i_2 \geq 1$. Then we have $|x_1 - x_2| < 1$ and $1 > |(x_1 - i_1) - (x_2 - i_2)| = |(x_1 - x_2) + (i_2 - i_1)| \geq |i_2 - i_1| - |x_1 - x_2| > 2 - 1 = 1$, a contradiction.

Thus we have $C(i_1, \dots, i_n, i_{n+1}) \subset D(i_1, \dots, i_n, i_{n+1})$ for each $(i_1, \dots, i_n, i_{n+1}) \in L$, and the open sets $D(i_1, \dots, i_n, i_{n+1})$ are disjoint for different $(i_1, \dots, i_n, i_{n+1}) \in L$. Then we have by the above inclusion $V(C(i_1, \dots, i_n, i_{n+1})) \leq V(D(i_1, \dots, i_n, i_{n+1}))$, and by the above disjointness $V(D(i_1, \dots, i_n, i_{n+1})) \leq V(C(i_1, \dots, i_n, i_{n+1}))$. We have that both $C(i_1, \dots, i_n, i_{n+1})$ and $D(i_1, \dots, i_n, i_{n+1})$ are relative interiors of convex bodies (in fact, polytopes) in $\text{lin } L$, namely of $\text{cl } C(i_1, \dots, i_n, i_{n+1})$ and $\text{cl } D(i_1, \dots, i_n, i_{n+1})$. From above, the first convex body is contained in the second one, while by the inequality $V(\text{cl } C(i_1, \dots, i_n, i_{n+1})) = V(C(i_1, \dots, i_n, i_{n+1})) \leq V(D(i_1, \dots, i_n, i_{n+1})) = V(\text{cl } D(i_1, \dots, i_n, i_{n+1}))$ the inclusion cannot be proper. Thus, the D-V cell of $(i_1, \dots, i_n, i_{n+1})$ in L , with respect to $\text{lin } L$, equals $\{(x_1, \dots, x_n, x_{n+1}) \in L \mid \max_{1 \leq j \leq n+1} (x_j - i_j) - \min_{1 \leq j \leq n+1} (x_j - i_j) \leq 1\}$. In particular, the D-V cell of $(0, \dots, 0, 0)$ in L , with respect to $\text{lin } L$, equals

$$\left\{ (x, \dots, x_n, x_{n+1}) \in \text{lin } L \mid \max_{1 \leq j \leq n+1} x_j - \min_{1 \leq j \leq n+1} x_j \leq 1 \right\}.$$

Recall the definition of $D(i_1, \dots, i_n, i_{n+1})$, in particular that of $D(0, \dots, 0, 0)$ and the considerations before it. These show that the D-V cell of $(0, \dots, 0, 0)$ in L with respect to $\text{lin } L$ equals $\{(x_1, \dots, x_n, x_{n+1}) \in \text{lin } L \mid \|x\|^2 \leq \|x + e_j - e_l\|^2 \ (1 \leq j \neq l \leq n+1)\}$. This set, P , say, is a convex polytope, with at most $n(n+1)/2$ facets. Observe, however, that P is invariant under the permutations of the coordinates x_1, \dots, x_n, x_{n+1} . This means that if, e.g., the inequality $\|x\|^2 \leq \|x + e_1 - e_2\|^2$ would not contribute a facet to P , then neither of the inequalities $\|x\|^2 \leq \|x + e_j - e_l\|^2$ would contribute one, so P would have no facets at all, which is impossible. This shows that each inequality $\|x\|^2 \leq \|x + e_j - e_l\|^2$, used for the definition of our set, contributes a facet to P .

Thus the D-V cell of $(0, \dots, 0, 0)$ with respect to L , in $\text{lin } L$, is a convex

polytope in $\text{lin } L$ with $n(n+1)/2$ facets, whose affine hulls are those affine hyperplanes in $\text{lin } L$, which are the perpendicular bisectors of the segments $[0, e_j - e_l]$ ($1 \leq j \neq l \leq n+1$). All these hyperplanes contain the respective points $(e_j - e_l)/2$, which lie on the boundary of the ball in $\text{lin } L$, with centre 0 and radius $1/\sqrt{2}$, and are actually tangent hyperplanes of this ball at the respective points $(e_j - e_l)/2$. Thus this polytope P , say, that evidently is a lattice space filler, is the polar, with respect to the unit ball in $\text{lin } L$ with centre 0, of the polytope $Q := \text{conv} \{e_j - e_l \mid 1 \leq j \neq l \leq n+1\}$, that is inscribed to the ball in $\text{lin } L$, with centre 0 and radius $\sqrt{2}$.

Now observe that Q is the difference body $S - S$ of the regular simplex S , of edge length $\sqrt{2}$ and with vertices $e_j \in \mathbb{R}^{d+1}$, introduced at the beginning of the proof (or we can say also with vertices $e_j - (e_1 + \dots + e_n + e_{n+1})/(n+1) \in \text{lin } L$). We have $V(S) = \sqrt{n+1}/n!$. However, we can calculate the volume of P as well. P is the D-V cell of 0 with respect to L , in $\text{lin } L$. The orthogonal projection of L to the $x_1 \dots x_n$ -coordinate plane is \mathbb{Z}^n , that has number density 1 in the $x_1 \dots x_n$ -coordinate plane. A unit normal vector of $\text{lin } L$ in \mathbb{R}^{n+1} is $(1/\sqrt{n+1}, \dots, 1/\sqrt{n+1}, 1/\sqrt{n+1})$. So by the above projection the volume of the image of a set is $\langle (1/\sqrt{n+1}, \dots, 1/\sqrt{n+1}, 1/\sqrt{n+1}), (0, \dots, 0, 1) \rangle = 1/\sqrt{n+1}$ times the volume of the original set. This means that the number density of L in $\text{lin } L$ is $1/\sqrt{n+1}$ times the number density of \mathbb{Z}^n in the $x_1 \dots x_n$ -coordinate plane, i.e., it is $1/\sqrt{n+1}$. In other words, $V(P)$, that equals the volume of a basic parallelotope in L , equals $\sqrt{n+1}$.

By [\[7\]](#), [\[32\]](#), Lemma (2.3), and [\[38\]](#), Theorem 1. we obtain

$$d_{n,n-1}(S) = \frac{V(((S-S)/2)^*)V(S)}{4^n \delta_L(((S-S)/2)^*)}.$$

From above we have $\delta_L(((S-S)/2)^*) = 1$. Moreover, $V((S-S)^*) = V(Q^*) = V(P) = \sqrt{n+1}$, and $V(S) = \sqrt{n+1}/n!$. These readily give the first equality. The second equality follows from Stirling's formula. \blacksquare

Proof of Theorem 3.5: Let $L \subset \mathbb{R}^n$ be a lattice, and $\lambda_1(B^n, L) \leq \dots \leq \lambda_n(B^n, L)$ its successive minima with respect to the convex body B^n . From Minkowski's theorem (cf., e.g., [\[21\]](#), p. 750) we have

$$\lambda_1(B^n, L) \cdots \lambda_n(B^n, L) \leq \frac{2^n D(L)}{\kappa_n / \delta_L(B^n)} = \frac{2^n \delta_L(B^n) D(L)}{\kappa_n}.$$

(Here $\kappa_n / \delta_L(B^n)$ is the smallest possible value of the absolute value of the determinant of a packing lattice of B^n .)

We have

$$\begin{aligned} \lambda_1(B^n, L) \cdots \lambda_k(B^n, L) &\leq (\lambda_1(B^n, L) \cdots \lambda_n(B^n, L))^{k/n} \leq \\ 2^k (\delta_L(B^n) D(L) / \kappa_n)^{k/n} &= 2^k (\delta_L(B^n) / \kappa_n)^{k/n} D(L)^{k/n}. \end{aligned}$$

Also by Gruber-Handbook [21], p. 750, we have k linearly independent vectors from L , with lengths $\lambda_1(B^n, L), \dots, \lambda_k(B^n, L)$ (actually the same holds with n rather than k). These span a k -dimensional sublattice, with absolute value of determinant in $(0, \lambda_1(B^n, L) \cdots \lambda_k(B^n, L)] \subset (0, 2^k (\delta_L(B^n) / \kappa_n)^{k/n} D(L)^{k/n}]$, hence $c_{n,k} \leq 2^k (\delta_L(B^n) / \kappa_n)^{k/n}$. The last inequality of the theorem follows from the estimate of $\delta_L(B^n)$ in § 1 and the formula for κ_n (and Stirling's formula).

For $k = 1$ we have equality in the first inequality of the theorem by the equation before the theorem, containing $c_{n,1}$.

It remained to show that for $2 \leq k \leq n - 1$ the first inequality of Theorem 2 is always strict. In fact, the first inequality in the second chain of inequalities in the proof is not sharp, unless $\lambda_1(B^n, L) = \dots = \lambda_n(B^n, L)$. Then the first inequality in the first chain of inequalities reduces to the inequality

$$\lambda_1(B^n, L)^n \leq \frac{D(L)}{2^{-n} \kappa_n / \delta_L(B^n)},$$

which is sharp only for a lattice similar to the lattice of a densest lattice packing of B^n . Now consider the n linearly independent vectors from L , chosen with lengths $\lambda_1(B^n, L), \dots, \lambda_n(B^n, L)$. If all of the first k vectors chosen with lengths $\lambda_1(B^n, L), \dots, \lambda_k(B^n, L)$ are not mutually orthogonal, then the parallelepiped spanned by them has a smaller volume than $\lambda_1(B^n, L) \cdots \lambda_k(B^n, L) = \lambda_1(B^n, L)^k$. So we will have a strict inequality. Since $\lambda_1(B^n, L) = \dots = \lambda_n(B^n, L)$, we could have chosen any k of them, in any order. If not all the n vectors are mutually orthogonal, we could have chosen the first and second ones not orthogonal, hence we have strict inequality. The same works if among all minimal vectors of L (of length $\lambda_1(B^n, L)$) there are some not orthogonal ones. So the only case that remains is that the lattice L has only n linearly independent minimum vectors, which are pairwise orthogonal. Thus the total number of minimum vectors of L is $2n$ (± 1 times the n linearly independent minimum vectors), while in any densest lattice packing of balls the number of minimum vectors is at least $n(n + 1)$ (Gruber-Lekkerkerker [23], p. 301). Hence the lattice L is not similar to the lattice of any densest lattice packing of B^n , a contradiction. ■

Proof of Proposition 3.6: Let $1 \leq l \leq n - k - 1$ be an integer, and let $\{e_1, \dots, e_n\}$ be a base of \mathbb{R}^n , such that $\text{lin}\{e_{n-k+1}, \dots, e_n\} = X$. Let the lattice L have as base $\{e_1, \dots, e_{n-k}, e_{n-k+1} + \lambda_1 e_1 + \dots + \lambda_l e_l, e_{n-k+2}, \dots, e_n\}$, where $1, \lambda_1, \dots, \lambda_l \in \mathbb{R}$ are linearly independent over the rationals, and let us consider the body lattice $\{x + rK \mid x \in L\}$, where $r \in (0, \infty)$. The union of this body lattice intersects each translate of X if and only if, for the projection π of \mathbb{R}^n to $\text{lin}\{e_1, \dots, e_{n-k}\}$ along X , we have that the projection of the body lattice $\{x + rK \mid x \in L\}$ covers $\text{lin}\{e_1, \dots, e_{n-k}\}$.

This projection equals $P(r) := \pi(\bigcup\{i_1 e_1 + \dots + i_{n-k} e_{n-k} + i_{n-k+1}(e_{n-k+1} + \lambda_1 e_1 + \dots + \lambda_l e_l) + i_{n-k+2} e_{n-k+2} + \dots + i_n e_n + rK \mid i_1, \dots, i_n \in \mathbb{Z}\}) = \bigcup\{i_1 e_1 + \dots + i_{n-k} e_{n-k} + i_{n-k+1}(\lambda_1 e_1 + \dots + \lambda_l e_l) + r\pi K \mid i_1, \dots, i_{n-k+1} \in \mathbb{Z}\}$. By the Theorem of Kronecker ([8], pp. 68-69) we have that the countably infinite set $S := \{i_1 e_1 + \dots + i_l e_l + i_{n-k+1}(\lambda_1 e_1 + \dots + \lambda_l e_l) \mid i_1, \dots, i_l, i_{n-k+1} \in \mathbb{Z}\}$ is dense in $\text{lin}\{e_1, \dots, e_l\}$. Then $P(r) = \bigcup\{S + r\pi K + i_{l+1} e_{l+1} + \dots + i_{n-k} e_{n-k} \mid i_{l+1}, \dots, i_{n-k} \in \mathbb{Z}\}$ (here $S + r\pi K$ is the Minkowski sum). We have $\text{lin}\{e_1, \dots, e_l\} + r \cdot \text{relint}(\pi K) \subset S + r\pi K \subset \text{lin}\{e_1, \dots, e_l\} + r\pi K$ (relint and later relbd meant with respect to the linear hull of the set in question). Here the first or third set is a relatively open, or closed, convex cylinder, with axis $\text{lin}\{e_1, \dots, e_l\}$ and base a relatively open, or closed, bounded convex set in $\text{lin}\{e_{l+1}, \dots, e_{n-k}\}$ (the relatively open/closed cylinder being the relative interior/closure of the other one). Denoting by ϱ the projection of $\text{lin}\{e_1, \dots, e_{n-k}\}$ to $\text{lin}\{e_{l+1}, \dots, e_{n-k}\}$, along $\text{lin}\{e_1, \dots, e_l\}$, we have that these bases are $\text{relint}\varrho(\pi K)$, or $\varrho(\pi K)$, respectively. Observe that $\pi K \subset \text{lin}\{e_1, \dots, e_{n-k}\}$ is a strictly convex body.

The intersection of $S + r\pi K$ with the common relative boundary of the above two cylinders is a small subset of this common boundary. Namely, it is the union of countably many translates of the set $[\text{rel bd}(\pi K)] \cap \varrho^{-1}[\text{rel bd}\varrho(\pi K)]$ (which is called the *shadow boundary, taken in* $\text{lin}\{e_1, \dots, e_{n-k}\}$, *of the convex body* $\pi K \subset \text{lin}\{e_1, \dots, e_{n-k}\}$ *with respect to illumination from the direction of* $\text{lin}\{e_1, \dots, e_l\}$). The restriction of ϱ to this set is injective, since πK is strictly convex; so this set is topologically an S^{d-2} . Any of these countably many translates is both nowhere dense in the common relative boundary of these cylinders, and has $(n - k - 1)$ -Hausdorff measure 0. There is an $r_0 > 0$ such that $\bigcup\{r\varrho(\pi K) + i_{l+1} e_{l+1} + \dots + i_{n-k} e_{n-k} \mid i_{l+1}, \dots, i_{n-k} \in \mathbb{Z}\} = \text{lin}\{e_{l+1}, \dots, e_{n-k}\}$ (i.e., $\bigcup\{\text{lin}\{e_1, \dots, e_l\} + r\pi K + i_{l+1} e_{l+1} + \dots + i_{n-k} e_{n-k} \mid i_{l+1}, \dots, i_{n-k} \in \mathbb{Z}\} = \text{lin}\{e_1, \dots, e_{n-k}\}$) holds if and only if $r \geq r_0$. Then for $r = r_0$ we have $P(r) \neq \text{lin}\{e_1, \dots, e_{n-k}\}$. However, for $r > r_0$ we have $P(r) \supset \bigcup\{\text{lin}\{e_1, \dots, e_l\} + r \cdot \text{relint}(\pi K) + i_{l+1} e_{l+1} + \dots + i_{n-k} e_{n-k} \mid i_{l+1}, \dots, i_{n-k} \in \mathbb{Z}\} = \text{lin}\{e_1, \dots, e_{n-k}\}$, hence $P(r) = \text{lin}\{e_1, \dots, e_{n-k}\}$.

Turning back to the body lattice $\{x + rK \mid x \in L\}$, we see that its union intersects each translate of X if and only if $r > r_0$. Dividing by r , we have that $\bigcup\{y + K \mid y \in L/r\}$ intersects each translate of K if and only if $r > r_0$. This is equivalent to the claim of the proposition. \blacksquare

Proof of Theorem 3.7: Let $L \subset \mathbb{R}^n$ be a lattice such that the lattice of unit balls $\{B^n + x \mid x \in L\}$ is k -impassable. We will estimate the density of this ball lattice from below.

Let $L_k \subset L$ be a k -dimensional sublattice with $|\det L_k| = D_k(L)$ and, hence, with $L \cap \text{lin } L_k = L_k$. Consider the orthogonal projection $\mathbb{R}^n \rightarrow (\text{lin } L)^\perp$, where $\dim(\text{lin } L)^\perp = n - k$. The image of L by this projection will be a lattice, $\Lambda \subset (\text{lin } L)^\perp$, say. We have

$$\begin{aligned} D(\Lambda) &= D(L)/|\det L_k| = D(L)/D_k(L) \geq D(L)/(c_{n,k}D(L)^{k/n}) = \\ &D(L)^{(n-k)/n} c_{n,k} \geq D(L)^{(n-k)/n} \cdot 2^{-k} (\kappa_n/\delta_L(B^n))^{k/n}, \end{aligned}$$

where the last inequality follows from Theorem 3.4.

The projection of the n -dimensional lattice of unit balls will be an $(n - k)$ -dimensional lattice of unit balls in $(\text{lin } L)^\perp$. Its density is $\kappa_{n-k}/D(\Lambda)$. If $\kappa_{n-k}/D(\Lambda) < \vartheta_L(B^{n-k})$, then the projected lattice cannot be a covering lattice for $(\text{lin } L)^\perp$. Hence there is an $x \in (\text{lin } L)^\perp \setminus \bigcup\{B^{n-k} + y \mid y \in \Lambda\}$, where in the last formula we mean by B^{n-k} the unit ball of $(\text{lin } L)^\perp$. Then the inverse image of x by the projection will be an affine k -plane in \mathbb{R}^n , which is a translate of $\text{lin } L$ disjoint to $\bigcup\{B^n + x \mid x \in L\}$, a contradiction.

Hence we have

$$\begin{aligned} \vartheta_L(B^{n-k}) &\leq \kappa_{n-k}/D(\Lambda) = \kappa_{n-k}D_k(L)/D(L) \leq \\ \kappa_{n-k}c_{n,k}/D(L)^{(n-k)/n} &\leq \kappa_{n-k} \cdot 2^k (\delta_L(B^n)/\kappa_n)^{k/n} / D(L)^{(n-k)/n}. \end{aligned}$$

Then we have for the density $\kappa_n/D(L)$ of the lattice $L \subset \mathbb{R}^n$ that

$$\begin{aligned} \kappa_n/D(L) &\geq \frac{\kappa_n \vartheta_L(B^{n-k})^{n/(n-k)}}{\kappa_{n-k}^{n/(n-k)} c_{n,k}^{n/(n-k)}} \geq \\ &\frac{\kappa_n \vartheta_L(B^{n-k})^{n/(n-k)}}{\kappa_{n-k}^{n/(n-k)} 2^{kn/(n-k)} (\delta_L(B^n)/\kappa_n)^{k/(n-k)}}. \end{aligned}$$

Since the first inequality of Theorem 3.4 was strict for $k \geq 2$, also the second inequality of Theorem 3.5 is strict for $k \geq 2$.

On the other hand, we have $d_{n,k} \geq d_{n,n-1} = k_n^2/(4^n \delta_L(B^{(n)})) = n^{-n} e^{O(n)}$ by § 1, (III) in § 2, and by Stirling's formula. ■

Proof of Corollary 3.9: We evaluate the second lower bound from Theorem 3.7 for $n = 2$, using $\delta_L(B^2) = \pi/\sqrt{12}$ and $\vartheta_L(B^1) = 1$.

The sharpness of the estimate for $n = 2$ follows from [38]. ■

Proof of Proposition 3.11: By (2.18) from the introduction there exists a constant n_0 such that for any $n \geq n_0$ and $n/\log_2 n \leq k \leq n - 1$ there exists a lattice packing $\{B^n + x \mid x \in L\}$ of unit balls that is k -impassable. Then $d_{n,k}$ is at most the density of this lattice packing, which is in turn at most $\delta_L(B^n)$. This proves the first inequality. For the second inequality cf. in the preliminaries. ■

Proof of Theorem 3.12: By [11], [41], [5] (the paragraph before Corollary 3, and Proposition 10), and [6] (pp. 147-148) a convex body, or a centrally symmetric convex body $K \subset \mathbb{R}^n$ can be included into an ellipsoid E such that $V(K)/V(E) \geq V(S^n)/V(B^n) = (n+1)^{(n+1)/2}/(\kappa_n n! n^{n/2})$ or $V(K)/V(E) \geq V(B^n)/V(C^n) = 2^n/(\kappa_n n!)$, where S^n , or C^n , are the regular simplex, or regular cross-polytope, inscribed to B^n , respectively. (Furthermore, the only cases of equality are that K is a simplex, or cross-polytope, respectively.)

Now let us include our convex body, or centrally symmetric convex body K , into such an ellipsoid E , as stated above. If for a point lattice $L \subset \mathbb{R}^d$ we have that the body lattice $\{K + x \mid x \in L\}$ is k -impassable, then also the body lattice $\{E + x \mid x \in L\}$ is k -impassable, with the quotient of the density of the mentioned lattice of translates of K and the density of the mentioned lattice of translates of E being $V(K)/V(E)$ (for which quantity we have the lower estimates given above). The second body lattice is a lattice of ellipsoids. If E were a ball, the density of this second body lattice would be at least $d_{n,k}$, by definition. However, $d_{n,k}(K)$ is invariant under affine transformations of K . So the density of the second body lattice is at least $d_{n,k}$, anyway.

Then the density of the first body lattice (of translates of K) is the product of the density of the second body lattice (for which the lower estimate $d_{n,k}$ holds) and the quotient of the densities of the first body lattice and the second body lattice (that equals $V(K)/V(E)$, for which we have the above given lower estimates). The product of these lower estimates gives the first inequality (both for the general and the centrally symmetric case). The second and third inequalities follow from Theorem 3.7.

On the other hand, we have $d_{n,k}(K) \geq d_{n,n-1}(K) > \kappa_n^2 \binom{2n}{n}^{-1} = e^{O(n)}/n^n$, by

lequa2.10
(117). ■

Proof of Theorem 3.16: A. By Reisner [41], for K a zonoid centred at 0, or a polar of a zonoid centred at 0, we have that Mahler's conjecture is true, i.e., $V(K)V(K^*) \geq 4^n/n!$. Then lequa2.5 (110) gives

$$d_{n,n-1}(K) = V(K)V(K^*)/[4^n\delta_L(K^*)] \geq V(K)V(K^*)/4^n \geq 1/n!.$$

For K a cross-polytope we have that K^* is a parallelotope, hence $\delta_L(K^*) = 1$, and so the second minimum in the theorem equals $1/n!$.

By Reisner [41], for K a zonoid centred at 0 we have $V(K)V(K^*) > 4^n/n!$, unless K is a parallelotope. For K a parallelotope centred at 0 we have that K^* is a cross-polytope. Then $\delta_L(K^*) < 1$, since else a lattice of translates of a cross-polytope would tile \mathbb{R}^n and so all its facets ought to be centrally symmetric (see Gruber-Lekkerkerker [23], p. 168), which is false for $n \geq 3$. Now observe that the expression for $d_{n,n-1}(K)$ is affine invariant and continuous in K , hence by Schneider [45], p. 60, Note 13, the first minimum in Theorem 3.16 exists (is attained). The same consideration proves that the second minimum in the theorem is attained only for a cross-polytope.

B. Analogously, for the bodies in the third and fourth minima, Mahler's conjecture is true, by Saint-Raymond [44], Théorème 25 and 28. Then an analogous consideration proves that this minimum is also equal to $1/n!$.

We turn to show that, in the third minimum, the minimum is attained only for cross-polytopes. For this aim, we have to recall a result of Meyer [40], Théorème 1.3 and 1.4, that establishes all cases of equality in the inequality $V(K)V(K^*) \geq 4^n/n!$, for $K \subset \mathbb{R}^n$ a convex body symmetric to all coordinate hyperplanes. All these cases of equality are obtained in the following way. Let the dimension $n \geq 1$ be fixed. We consider \mathbb{R}^n , with the standard base $\{e_1, \dots, e_n\}$. We will work with coordinate subspaces of \mathbb{R}^n . We begin with defining 0-symmetric convex bodies in all 1-dimensional coordinate subspaces of \mathbb{R}^n , namely the segments $[-e_i, e_i]$. We proceed by induction. Let us have a set of coordinate subspaces $\{X_j\}$ in \mathbb{R}^n , with pairwise intersections $\{0\}$, and together spanning \mathbb{R}^n , and in each X_j let us have an 0-symmetric convex body K_j . If there are at least two such X_j -s, we pick two of them, and replace this pair by their (direct) sum, and replace the corresponding two K_j -s either by their (direct) sum, or by the convex hull of their union. We end when we have only one X_j , and the corresponding K_j will be the body constructed this way. (The normed spaces corresponding to these convex bodies

are called *Hanner-Hansen-Lima spaces*, because these authors investigated their properties.) Clearly, all bodies constructed this way are polytopes symmetric w.r.t. all coordinate hyperplanes. Furthermore, the polar of such a polytope is such a polytope as well.

We are going to show that, unless K is a cross-polytope (i.e., K^* is a parallelotope), we have $\delta_L(K^*) < 1$. For this, like in **A**, it will suffice to show that K^* has a facet which is not centrally symmetric. We will use induction for n .

We will use that also $B := K^*$ is obtained by the above construction. In the last step of the construction, we will have two coordinate subspaces X_1, X_2 , and convex polytopes B_1, B_2 in them.

If B is the direct sum of B_1 and B_2 , and is not a parallelotope, then one B_j is not a parallelotope. Then, by the induction hypothesis, B_j has a facet F_j , say, which is not centrally symmetric. Then B has a facet $F_j \oplus B_{2-j}$ that is not centrally symmetric either.

Now let B be the convex hull of the union of B_1 and B_2 . By turning to the polar bodies, we see that the vertices of B^* are the direct sums of any vertex of B_1^* and any vertex of B_2^* . Hence, the facets of B are the convex hulls of the unions of any facets of B_1 and B_2 . Now recall that $n \geq 3$. (We remark that the case $n = 2$ is anyway trivial, since the direct sum and the convex hull of the union of B_1 and B_2 both are parallelograms.)

Let therefore F_j be a facet of B_j , and consider the facet $F = \text{conv}(F_1 \cup F_2)$ of B . Observe that $2 \leq n - 1 = \dim F = \dim F_1 + \dim F_2 + 1$. Hence one of $\dim F_j$ is positive. Let us consider $\text{aff } F$. Observe that $(\text{aff } F_1) \cap (\text{aff } F_2)$ is a subset of the intersection of the corresponding coordinate hyperplanes, i.e., of $\{0\}$. However, $F_j \ni 0$, so $(\text{aff } F_1) \cap (\text{aff } F_2)$ is empty. There is, up to translations, just one hyperplane H in $\text{aff } F$ that is parallel to both $\text{aff } F_j$. The boundary hyperplanes of the supporting strip of F in $\text{aff } F$, parallel to H , intersect F in F_1 and F_2 , respectively. If F would be centrally symmetric, the affine hulls of these intersections would be translates of each other. So, if we consider their translates containing 0, these would coincide. However, these translates have intersection $\{0\}$. So these translates would be $\{0\}$, so both would have dimension 0. This, however, contradicts the fact that one of $\dim F_j$ is positive.

C. Lastly, for $n \leq 8$, also for the bodies in the fourth and fifth minima Mahler's conjecture is true, by [Lopez-Reisner \[35\]](#). Even, they prove that the only cases of

equality are of the form described in **B**. Then the considerations of **B** show that the only case, when in

$$d_{n,n-1}(K) = V(K)V(K^*)/[4^n \delta_L(K^*)] \geq V(K)V(K^*)/4^n \geq 1/n!$$

we have equality at both inequalities, is when K is a cross-polytope.

This shows that the fifth minimum in the theorem is greater than $1/n!$, and the sixth minimum is $1/n!$, and is attained only for K a cross-polytope. ■

Proof of Theorem 3.18: The considerations after Conjecture 3.17 give, for the minimum in the theorem, the lower estimate

$$\left[2^n / \binom{2n}{n}\right] \cdot \min\{V(K)V(K^*) \mid K \subset \mathbb{R}^n \text{ is an 0-symmetric convex body}\}.$$

Then we apply the theorem of Reisner [41] used in the proof of Theorem 3.16.

For K a simplex, Proposition 3.1 gives $d_{n,n-1}(K) = 2^n(n+1)/n!$. It remains to show that, for K a simplex, $((K - K)/2)^*$ is a zonoid. We are going to show that it is a zonotope.

Like in the proof of Proposition 3.1, we may suppose that K equals a regular simplex with edge length $\sqrt{2}$, which we denote by S . Further we use the notations from the proof of Proposition 3.1. There it was proved that the D-V cell of $(0, \dots, 0, 0)$ with respect to L , in $\text{lin } L$, is the polar of the polytope $Q = \text{conv}\{e_j - e_l \mid 1 \leq j \neq l \leq n+1\} = S - S$, with respect to the unit ball in $\text{lin } L$ with centre 0. So the above D-V cell is $(S - S)^*$. By Vegh [48], this D-V cell is the orthogonal projection of the D-V cell of $(0, \dots, 0, 0)$ with respect to \mathbb{Z}^{n+1} , in \mathbb{R}^{n+1} , i.e., of the unit cube $C = [-1/2, 1/2]^{n+1}$. (Considering the nearest neighbours of $(0, \dots, 0, 0)$ in \mathbb{Z}^n gives that this D-V cell is contained in C and also has volume 1, so they are equal.) Here C is a zonotope, hence its orthogonal projection $(S - S)^*$ is a zonotope, too. And $((S - S)/2)^* = 2(S - S)^*$ is a zonotope as well. ■

Proof of Theorem 3.22: The equality follows from the inequality $d_{2,1}(K) \leq d_{2,1}((K - K)/2)$, cf. Makai [38], Theorem 4. Now we prove the inequality for K centrally symmetric.

By ^{Makai-1978}[38], Theorem 5, we have, for centrally symmetric K ,

$$d_{2,1}(K) = \frac{\pi^2}{16 \min\{\delta_L(K')\}},$$

where the minimum is taken for all centrally symmetric convex bodies $K' \subset \mathbb{R}^2$. This minimum satisfies

$$\min\{\delta_L(K')\} \geq 0.8926 \dots,$$

cf. ^{Tammela}[47]. These two inequalities imply our theorem. ■

Proof of Theorem 3.25: The density of the body lattice $\{K + x \mid x \in L\}$ is at most $\delta_L(K)$, that is smaller than $d_{n,k}(K)$. Hence there exists an affine k -plane disjoint to $\bigcup\{K + x \mid x \in L\}$. Even, we may inflate K by a factor $(d_{n,k}(K)/\delta_L(K))^{1/n}$, and still we find an affine k -plane A_k disjoint to $\bigcup\{\text{int}((d_{n,k}(K)/\delta_L(K))^{1/n}K + x \mid x \in L\}$. Inflation of K by a factor $(d_{n,k}(K)/\delta_L(K))^{1/n}$ can be written also as $K + [(d_{n,k}(K)/\delta_L(K))^{1/n} - 1]K$. Then disjointness of $\{\text{int}(K + [(d_{n,k}(K)/\delta_L(K))^{1/n} - 1]K) + x \mid x \in L\}$ and A_k is equivalent to disjointness of $\{K + x \mid x \in L\}$ and $\text{int}(A_k + [(d_{n,k}(K)/\delta_L(K))^{1/n} - 1](-K))$. ■

Proof of Proposition 3.27: By Theorem 3.7, we have $d_{4,1} > 25\pi^2/256$. We have $\delta_L(B^4) = \pi^2/16$ (^{Rogers}[42], p. 3), which is smaller than $25\pi^2/256$ ($\leq d_{4,1}$). Then, by Theorem 3.25, applied to $B^4 \subset \mathbb{R}^4$, there is an open, both-way infinite cylinder with base a 3-ball of radius $(d_{4,1}/\delta_L(B^4))^{1/4} - 1 > [(25\pi^2/256)/(\pi^2/16)]^{1/4} - 1 = \sqrt{5}/2 - 1$, which is disjoint to our lattice packing of closed unit balls. ■

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